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Superposition of the Abelian Yang–Mills potentials

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Abstract. It is shown that Abelian Yang–Mills potentials can be superposed not only in the parallel case in isospin space. A necessary and sufficient condition for superposition of two such potentials is given. Their sum has a non-trivial (i.e. non-Maxwellian) form. Some suggestions concerning physical relevance are included.

1. Introduction

The majority of classical solutions of Yang–Mills (YM) equations are sourceless or generated by single source (quark) potentials. But it is well known that hadrons consist of two or more quarks. Hence potentials which have two or more sources are physically more interesting.

The simplest such potential is the Coulomb potential with sources in commuting directions (Mandula 1976). If the sources are not in commuting directions the potential is not Maxwellian. The field of such a system of sources has non-Abelian properties (Arodź 1978).

In this paper it is considered under what conditions the system of Abelian sources generates the field which is the same as the superposition of the potentials of separate sources. A necessary and sufficient condition for two Maxwell-type potentials to be superposed is found.

It will be shown that in this way new, non-trivial (i.e. non-Maxwellian) potentials can be constructed. However they possess some Abelian properties. These potentials belong to the wider class of YM potentials (generalised Maxwell-type (GMT) potentials) which will be defined later. The superposition of three (or more) potentials also belongs to the class of GMT potentials. Some of their properties are also mentioned.

To the best of our knowledge the problem of the superposition of potentials in YM theory has not been investigated yet. Only the possibility of the superposition of the Coulomb-type potentials with parallel sources (in isospin space) has been noticed (Mandula 1976).

In the last section we speculate on the physical relevance of our results.

In our approach superposition means superposition of potentials and sources. If quarks could be superposed without any dramatic change in potentials (fields) it would mean that there is no sign of confinement at the classical level.

We shall restrict our discussion to the SU(2) gauge group but the results obtained can be extended for arbitrary semi-simple and compact Lie groups.

YM equations have the form

$$\partial^{\mu}\boldsymbol{F}_{\mu\nu}+\boldsymbol{A}^{\mu}\times\boldsymbol{F}_{\mu\nu}=\boldsymbol{j}_{\nu} \tag{1}$$

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where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu. \tag{1'}$$

Here, as usual, the vectors are in isospin space and the Greek letters are Minkowski indices.

These equations are nonlinear. This is the main difficulty in gauge theory and it destroys the superposition principle.

For Maxwell-type potentials i.e. those of the form

$$A_\mu(x) = A\alpha_\mu(x) \tag{2}$$

(A is a constant isovector) equation (1) reduces to the linear, Maxwell-type equation

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = j_\nu \tag{3}$$

(Maxwell 'type' because it differs from Maxwellian by an additional isospin index).

The nonlinear terms in (1), which vanish in that case, are

$$2A_\mu \times \partial^\mu A_\nu + \partial^\mu A_\mu \times A_\nu - A^\mu \times \partial_\nu A_\mu + A^\mu \times (A_\mu \times A_\nu) = 0. \tag{4}$$

It should be stressed that conditions (3) and (4) are not gauge invariant and because of this we do not fix a gauge.

2. Superposition of the Maxwell-type potentials

We shall consider two Maxwell-type potentials of the form (2)

$$A_\mu(x) = A\alpha_\mu(x) \qquad B_\mu = B\beta_\mu(x) \tag{5}$$

which fulfil reduced field equations

$$\Box A_\mu + \partial_\mu \partial^\nu A_\nu = j^A_\mu \tag{6a}$$

$$\Box B_\mu + \partial_\mu \partial^\nu B_\nu = j^B_\mu. \tag{6b}$$

In general, the sum of any two type (5) potentials does not satisfy field equations (1) with the source

$$j_\mu = j^A_\mu + j^B_\mu \tag{7}$$

because of the nonlinear term (4). This problem is clarified in the theorem below.

Theorem. Two Maxwell-type solutions (5) of the YM equations (which in this case reduce to equations (6)) can be superposed and their sum satisfies (1) with the source term (7) if and only if one of the three conditions is satisfied:

(i) $A \times B = 0$. This is a trivial case when sources are in the parallel isospin directions (Mandula 1976).

(ii) $A \times B \neq 0$ and vectors α_μ, β_μ are parallel in Minkowsi space i.e.

$$\beta_\mu(x) = \varphi(x)\alpha_\mu(x) \tag{8}$$

where $\alpha_\mu(x)$ and arbitrary scalar function $\varphi(x)$ must satisfy the equations

$$\alpha_\mu \alpha^\mu \partial_\nu \varphi - \alpha^\mu \alpha_\nu \partial_\mu \varphi = 0. \tag{9}$$

Then the potential obtained as a result of superposition is

$$C_\mu(x) = (A + \varphi(x)B)\alpha_\mu(x). \tag{10}$$

(iii) $\boldsymbol{A}\times\boldsymbol{B}\neq 0$ and $\beta_\mu(x)\neq\varphi(x)\alpha_\mu(x)$. Then the only possibility is $\alpha_\mu\alpha^\mu=\beta_\mu\beta^\mu=\alpha_\mu\beta^\mu=0$ and

$$2\alpha^\mu(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu)-2\beta_\mu\partial^\mu\alpha_\nu+\beta_\nu\partial^\mu\alpha_\mu-\alpha_\nu\partial^\mu\beta_\mu=0. \tag{11}$$

To prove the above theorem it should be noted that the potentials of the form (5) can be superposed if and only if their sum satisfies the linearisation condition (4). The essential point is that after this substitution the left-hand side of (4) is a linear combination of three constant isovectors:

$$\boldsymbol{A}\times\boldsymbol{B}, \qquad \boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B}) \qquad \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{B}).$$

For $\boldsymbol{A}\times\boldsymbol{B}\neq 0$ this combination is equal to zero if and only if all coefficients vanish. So we obtain three equations instead of (4). Two of them can be easily solved and their only solutions are $\beta_\mu=\varphi\alpha_\mu$ or $\alpha_\mu\alpha^\mu=\beta_\mu\beta^\mu=\alpha_\mu\beta^\mu=0$. Having substituted these solutions in the third equation we obtain (9) or (11) respectively.

Here a further comment is necessary. Field equations (6) are additional restrictions on α_μ, β_μ or α_μ, φ in case (ii). For special sources and potentials they may be incompatible with (8) and (9). But we did not specify the sources j^A_μ and j^B_μ—they are quite arbitrary.

Because of it equation (6) can be viewed as a constraint on sources not on potentials. It is also possible to consider conditions (i), (ii) and (iii) as purely kinematic.

We conclude that restrictions on sources are necessary to superpose quarks and not to change their potentials.

Now we shall examine condition (ii) in detail. It is satisfied if we change it for a stronger one

$$\alpha_\mu\partial_\nu\varphi-\alpha_\nu\partial_\mu\varphi=0. \tag{12}$$

Hence we obtain explicitly a special class of potentials which can be superposed with the potential (8)

$$\alpha_\mu(x)=f(x)\partial_\mu\varphi(x) \tag{13}$$

where $f(x)$ is an arbitrary scalar function.

In the trivial case, $f(x)=$ constant, (13) means that the potential is pure gauge. The second class of solutions to (9) is α_μ and φ which satisfy

$$\alpha_\mu\alpha^\mu=0 \qquad \text{and} \qquad \alpha_\mu\partial^\mu\varphi=0. \tag{14}$$

Solution (10) satisfies condition (4) and in this sense it has Abelian properties. However it can be noticed that the isospin of the potential obtained and source $j_\mu(x)$ is restricted to the plane perpendicular to the isovector $\boldsymbol{A}\times\boldsymbol{B}$. Maxwell-type potentials and sources have isospin restricted to one direction only. Hence the potential (10) is an essential generalisation of the Maxwell-type potentials.

The theorem given above can be generalised in the case of the superposition of the three potentials of the form

$$\boldsymbol{A}\alpha_\mu+\boldsymbol{B}\beta_\mu+\boldsymbol{C}\gamma_\mu. \tag{15}$$

The isospin of such a potential (and its source) is not restricted to the plane as in case (ii), but this is a much more complicated problem and will not be discussed here.

As a simple example we shall apply conditions (ii) and (iii) to static potentials of the form

$$\alpha_\mu(x)=\delta_{\mu 0}\alpha(\boldsymbol{x}) \qquad \beta_\mu(x)=\delta_{\mu 0}\beta(\boldsymbol{x}). \tag{16}$$

Ansatz (16) reduces equations (9) and (11) to

$$\alpha^2(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})=0 \tag{9'}$$

$$2\alpha(\boldsymbol{x})\nabla\beta(\boldsymbol{x})=0. \tag{11'}$$

These equations have only trivial, constant solutions. Hence we conclude that no potentials of the form (16) can be superposed other than those parallel in isospin space or with the same functional dependence on x: $\beta_\mu(x)=\text{constant}\,\alpha_\mu(x)$. In both cases after superposition a trivial (i.e. Abelian) potential is obtained.

It is interesting to notice that the time dependence of quark fields was suggested in a different context (Magg 1978).

3. Generalised Maxwell-type potentials

Potential (10) is a special case of the generalised Maxwell-type (GMT) potential whose form is

$$\boldsymbol{A}_\mu(x)=\boldsymbol{A}(x)\lambda_\mu(x). \tag{17}$$

These potentials have many simple properties. Some of them are listed below:

(i) linearisation condition (4) is reduced to

$$\lambda_\mu\lambda_\nu\boldsymbol{A}\times\partial^\mu\boldsymbol{A}-\lambda_\mu\lambda^\mu\boldsymbol{A}\times\partial_\nu\boldsymbol{A}=0 \tag{18}$$

(ii) condition (18) is equivalent to the vanishing of the field current i.e.

$$\boldsymbol{j}_\nu^{\text{field}}\equiv\boldsymbol{A}^\mu\times\boldsymbol{F}_{\mu\nu}=0 \tag{19}$$

(iii) none of the GMT potentials possesses monopole sources

$$\partial^k\boldsymbol{B}_k=0 \tag{20}$$

where

$$\boldsymbol{B}_k\equiv\tfrac{1}{2}\varepsilon^{kmn}\boldsymbol{F}_{mn}.$$

For any two arbitrary GMT potentials the condition for superposition has a more complicated form and we cannot solve it. But solving YM equations (1) in GMT form (17) is easier if we restrict our considerations to potentials which also satisfy condition (18). It is a set of partial differential equations of the first order.

As a solution of (18) we can easily obtain, for example, a non-Abelian plane wave (Coleman 1977) if we choose simply

$$\lambda_0=\lambda_3=1 \qquad \lambda_1=\lambda_2=0.$$

4. Final remarks

As has been shown (Wu and Yang 1975) there exists in YM theory a copying phenomenon i.e. different, gauge non-equivalent potentials can give the same field strength tensors $\boldsymbol{F}_{\mu\nu}$. It has been emphasised (Deser and Wilczek 1976) that this phenomenon is related to the Abelian properties of the potentials and is of considerable importance in quantum theory (Halpern 1978). These arguments imply that Abelian-type or GMT potentials are more important in gauge theory than was previously supposed.

A simple calculation shows that the trivial Maxwell-type and type (10) potentials satisfy the necessary condition for the existence of copying phenomena

$$[\boldsymbol{F}_{\mu\nu}\cdot(\boldsymbol{F}_{\alpha\beta}\times\boldsymbol{F}_{\sigma\rho})][{}^*\boldsymbol{F}^{\mu\nu}\cdot({}^*\boldsymbol{F}^{\alpha\beta}\times{}^*\boldsymbol{F}^{\sigma\rho})]=0 \tag{21}$$

where the asterisk means dual conjugation.

It should be stressed that conditions (i), (ii) and (iii) of § 2 are rather strong. Sources j^A_μ, j^B_μ cannot be arbitrary because in special cases these conditions can be incompatible with field equations (1). It indicates that for some sources the field after superposition has to change.

This conclusion can be regarded from two different points of view. From the one point it is possible that the field of real sources (quarks) after superposition is dramatically changed (confinement?). On the other hand it is possible (at least in the high-energy region) that sources and fields with additive properties are dominant. We hope that our results in this region can be useful.

As a next step restrictions on sources (implied by field equations (6)) should be investigated.

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